

Phase Structure of the Interacting Vector Boson Model

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The two-fluid Interacting Vector Boson Model (IVBM) with the $U(6)$ as a dynamical group possesses a rich algebraic structure of physical interesting subgroups that define its distinct exactly solvable dynamical limits. The classical images corresponding to different dynamical symmetries are obtained by means of the coherent state method. The phase structure of the IVBM is investigated and the following basic phase shapes, connected to a specific geometric configurations of the ground state, are determined: spherical, $U_p(3) \otimes U_n(3)$, γ -unstable, $O(6)$, and axially deformed shape, $SU(3) \otimes U_T(2)$. The ground state quantum phase transitions between different phase shapes, corresponding to the different dynamical symmetries and mixed symmetry case, are investigated.

PACS number(s): 21.60.Ev, 21.60.Fw, 21.10.Re

I. INTRODUCTION

The phase structure of quantum many-body systems has been a subject of great experimental and theoretical interest in the last years. The introduction of the concept of critical point symmetry [1] has recalled the attention of the community to the topic of quantum phase transitions in nuclei. Different models have been used to describe the quantum phase transitions in different many-body systems, such as atomic nuclei [2], molecules [3],[4], atomic clusters [5], and finite polymers. Among these models, those based upon algebraic Hamiltonians play an important role.

There are many approaches which allow the association of a certain geometry to any abstract algebra, but for algebraic models, this can be achieved with the theory of the coherent states [6],[7],[8],[9]. The expectation value of the Hamiltonian in the ground coherent state is referred to as its classical limit. The classical limit of quantum systems is one of the oldest problems in quantum mechanics. This problem appears whenever one formulates a theory in terms of quantum variables and wishes to interpret it in terms of classical (geometrical) variables. The method of coherent (or intrinsic) states provides a prescription for translating algebraic operators into canonical phase-space coordinates, thereby allowing algebraic models of nuclear structure and dynamics to be interpreted from the perspective of their corresponding classical limits. Of particular relevance for the present study is the construction of the potential energy or potential functions [2],[3].

A nice feature of the algebraic models is the occurrence of phases connected to a specific geometric configurations of the ground state, which arise from the occurrence of different dynamical symmetries. The study of the ground state energy as a function of an appropriately chosen parameter, called control parameter, shows a transition between the different phases. These phase transitions are referred to as ground-state or quantum phase transitions and have been widely investigated in the last years (e.g., a

review article [10]). Since these transitions are between different shapes, they are sometimes termed as "shape transitions".

An important aspect of the study of phase transitions is the construction of the phase diagram (structure). In this respect it is interesting to see what is the phase structure of the two-fluid Interacting Vector Boson Model (IVBM) [11]. Thus, it is the purpose of the present paper to investigate what are the different phase shapes which might occur within the framework of the IVBM. The first step in the construction of the phase diagram is the identification of all possible dynamical symmetries of the system. In IVBM there are several dynamical symmetries which will be considered in next sections. It should be shown that there exist three distinct shapes corresponding to the three dynamical symmetries of IVBM: (1) spherical shape, $U_p(3) \otimes U_n(3)$, (2) γ -unstable shape, $O(6)$, and (3) axially deformed shape, $SU(3) \otimes U_T(2)$.

II. THE ALGEBRAIC STRUCTURE GENERATED BY THE TWO VECTOR BOSONS

The algebraic structure of IVBM [11],[12] is realized in terms of creation and annihilation operators of two kinds of vector bosons $u_m^\dagger(\alpha)$, $u_m(\alpha)$ ($m = 0, \pm 1$), which differ in an additional quantum number $\alpha = \pm 1/2$ (or $\alpha = p$ and n)—the projection of the T -spin (an analogue to the F -spin of IBM-2). We consider these two bosons just as building blocks or "quarks" of elementary excitations (phonons) rather than real fermion pairs, which generate a given type of symmetry. In this regard, the s and d bosons of the IBM-1 can be considered as bound states of elementary excitations generated by two vector bosons. Thus, we assume that it is the type of symmetry generated by the bosons which is of importance for the description of the collective motions in nuclei rather than the tensorial or fermionic nature of these bosons.

The number preserving bilinear products of the creation and annihilation operators of the two vector bosons generate the boson representations of the unitary group

$U(6)$ [11],[12]:

$$A_M^L(\alpha, \beta) = \sum_{k,m} C_{1k1m}^{LM} u_k^\dagger(\alpha) u_m(\beta), \quad (1)$$

where C_{1k1m}^{LM} , which are the usual Clebsch-Gordan coefficients for $L = 0, 1, 2$ and $M = -L, -L+1, \dots, L$, define the transformation properties of (1) under rotations. We will also use the notations $u_m^\dagger(\alpha = 1/2) = p_m^\dagger$ and

$$u_m^\dagger(\alpha = -1/2) = n_m^\dagger.$$

In the most general case the two-body model Hamiltonian should be expressed in terms of the generators of the group $U(6)$. In some special cases the Hamiltonian can be written in terms of the generators of different subgroups of $U(6)$. The $U(6)$ group contains the following chains of subgroups [11],[12]:

$$\begin{array}{ccccc} & & U(6) & & \\ & \swarrow & \downarrow & \searrow & \\ U_p(3) \otimes U_n(3) & & O_\pm(6) & & U(3) \otimes U_T(2) \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ SO_p(3) \otimes SO_n(3) & & \overline{SU_\pm(3)} \otimes SO(2) & & SU_\pm(3) \otimes SO_T(2) \\ & \searrow & \downarrow & \swarrow & \\ & & SO(3) & & \end{array} \quad (2)$$

As can be seen, the IVBM has a rich enough algebraic structure of subgroups. Each of these dynamical symmetries will correspond to a certain possibly different shape phase. Now we are going to briefly enumerate these algebras, their generators and some of their Casimir operators relevant to the present work.

A. The $U_p(3) \otimes U_n(3)$ chain

(a) $U_p(3) \otimes U_n(3)$ algebra. It consists of two sets of commuting operators:

$$N_p = \sqrt{3}(p^\dagger \times p)^{(0)}, \quad (3)$$

$$L_M^p = \sqrt{2}(p^\dagger \times p)_M^{(1)}, \quad (4)$$

$$Q_M^p = \sqrt{2}(p^\dagger \times p)_M^{(2)}. \quad (5)$$

and

$$N_n = \sqrt{3}(n^\dagger \times n)^{(0)}, \quad (6)$$

$$L_M^n = \sqrt{2}(n^\dagger \times n)_M^{(1)}, \quad (7)$$

$$Q_M^n = \sqrt{2}(n^\dagger \times n)_M^{(2)}. \quad (8)$$

The $SU_\tau(3)$ ($\tau = p, n$) algebra is obtained by excluding the number operator N_τ , whereas the angular momentum algebra $SO_\tau(3)$ is generated by the generators L_M^τ only.

For the decomposition of fully symmetric irreducible representation (IR) $[N]_6$ of $U(6)$ into the IR's of $U_p(3) \otimes U_n(3)$, we have [13]:

$$[N]_6 = \sum_{m=0}^N [N-m]_3 \otimes [m]_3, \quad (9)$$

or in Elliott notations

$$[N]_6 = \sum_{m=0}^N (\lambda_p = N-m, \mu_p = 0) \otimes (\lambda_n = m, \mu_n = 0). \quad (10)$$

For the decomposition $SU_\tau(3) \supset SO_\tau(3)$, the standard reduction rules take place [13]

$$K = \min(\lambda, \mu), \min(\lambda, \mu) - 2, \dots, 0 \quad (1)$$

$$L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, \dots, 0 \quad (1); K = 0$$

$$L = K, K+1, \dots, K + \max(\lambda, \mu); K \neq 0. \quad (11)$$

The linear Casimir operators are simply

$$C_1[U_p(3)] = N_p \quad (12)$$

and

$$C_1[U_n(3)] = N_n. \quad (13)$$

The quadratic Casimir operator of the one-fluid $U_\tau(3)$ algebra can be expressed in the following multipole form

$$C_2[U_\tau(3)] = N_\tau(N_\tau + 3), \quad (14)$$

and that of $SU_\tau(3)$ as

$$C_2[SU_\tau(3)] = \frac{4}{3} Q^\tau \cdot Q^\tau + \frac{1}{2} L^\tau \cdot L^\tau. \quad (15)$$

(b) $O_p(3) \otimes O_n(3)$ algebra. This algebra is determined by the operators (4) and (7).

B. The $SU(3) \otimes U_T(2)$ chain

The $SU(3) \otimes U_T(2)$ algebra also consists of two commuting sets of operators:

(a) $U_T(2)$ algebra. It is defined by the operator of a number of particles

$$N = N_p + N_n \quad (16)$$

and the "T-spin" operators T_m^1 , ($m = 0, \pm 1$) introduced through

$$T_1^1 = \sqrt{\frac{3}{2}} A^0(p, n), \quad T_{-1}^1 = -\sqrt{\frac{3}{2}} A^0(n, p) \quad (17)$$

$$T_0^1 = -\frac{\sqrt{3}}{2} [A^0(p, p) - A^0(n, n)]. \quad (18)$$

The above operators T_m^1 ($m = 0, \pm 1$) commute with N . Thus (17) and (18) define the subalgebra $su_T(2) \subset u_T(2)$. These operators play an important role in the consideration of the nuclear system as composed by two interacting (proton and neutron) subsystems. At fixed N the "T-spin" T takes the values $T = N/2, N/2 - 1, \dots, 0$ or 1 , respectively. The IR's of the subgroup $SO_T(2) \subset SU_T(2)$ generated by T_0^1 are determined standardly as follows: $-T, -T + 1, \dots, T$. The second-order Casimir operator of $U_T(2)$ is

$$C_2[U_T(2)] = \frac{4}{3} T^2 + \frac{1}{3} N^2$$

with eigenvalue $\frac{4}{3} T(T + 1) + \frac{1}{3} N^2$.

(b) $U(3)$ algebra. It consists of the operators which are a sum of the p - and n -boson subsystem operators: the total number of bosons N (16), the angular momentum operator $L_M = L_M^p + L_M^n$ and the components of the "truncated" (or Elliott's) quadrupole operator

$$Q_M = Q_M^p + Q_M^n. \quad (19)$$

The operators L_M and Q_M commute with N and define the subalgebra $su(3) \subset u(3)$.

The second-order Casimir operator of $U(3)$ is

$$C_2[U(3)] = \frac{1}{6} Q^2 + \frac{1}{2} L^2 + \frac{1}{3} N^2 = \frac{1}{2} N^2 + 2T^2 + N, \quad (20)$$

where

$$Q^2 = 6 \sum_{M, \alpha, \beta} (-1)^M A_M^2(\alpha, \alpha) A_M^2(\beta, \beta).$$

The $SU(3)$ Casimir is simply

$$K_3 = \frac{1}{6} Q^2 + \frac{1}{2} L^2 \quad (21)$$

and its eigenvalue is $(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$. It should be pointed that the groups $U_T(2)$ and $U(3)$ are mutually complementary in the sense that the eigenvalues of $C_2[U(2)]$ are completely determined by the eigenvalues of $C_2[U(3)]$. This is due to the relation

$$C_2[U(3)] = \frac{3}{2} C_2[U_T(2)] + N. \quad (22)$$

This means that the representations of $U(3)$ and $U_T(2)$ can be labeled by the same quantum numbers (for instance, the number of quasiparticles N and the "T-spin" T).

With the help of standard group-theoretical methods [13] one obtains the quantum numbers $[E_1, E_2, E_3]_3 \times [\varepsilon_1, \varepsilon_2]_2$ labelling the IR of the direct product $U(3) \otimes U_T(2)$ belonging to a given IR of $U(6)$. In the case of a fully symmetric IR $[N]_6$ of $U(6)$ the decomposition is

$$[N]_6 = \sum_{i=0}^{<\frac{N}{2}>} [N - i, i, 0]_3 \times [N - i, i]_2, \quad (23)$$

where $<\frac{N}{2}> = \frac{N}{2}$ if N is even and $\frac{N-1}{2}$ if N is odd. In fact this means that $[N]_6$ decomposes into $SU(3)$ multiplets with $(\lambda, \mu) = (N - 2i, i)$, $i = 1, 2, \dots, <\frac{N}{2}>$. Each of these multiplets is characterized by a "T-spin" $T = \frac{\lambda}{2}$. The decomposition of any IR (λ, μ) of $SU(3)$ into the $SO(3)$ representations L is given by (11).

C. The $O_{\pm}(6)$ chain

(a) $O(6)$ algebra. The $O_+(6)$ algebra is spanned by the following operators:

$$A_M^1(p, p) \equiv L_M^p, \quad (24)$$

$$A_M^1(n, n) \equiv L_M^n, \quad (25)$$

$$G_{ij}^{(+)} = p_i^\dagger n_j + n_i^\dagger p_j. \quad (26)$$

An alternative $O_-(6)$ algebra can be defined with the generators

$$G_{ij}^{(-)} = i(p_i^\dagger n_j - n_i^\dagger p_j) \quad (27)$$

instead of $G_{ij}^{(+)}$. Both the $O_+(6)$ and $O_-(6)$ algebras have the same eigenspectrum but differ through phases in the wave functions. They are related by a transformation that is a special case of a wider class of transformations known as inner automorphisms. It is known that representation theory does provide all of the embeddings, but it does not provide all of the dynamical symmetries [14]. Indeed, the inner automorphisms can provide new dynamical symmetry limits, sometimes referred as to "hidden" [14] or "parameter" symmetries [15].

The second order Casimir operator of $O(6)$ is given by

$$C_2[O(6)] = 3(G^\pm \times G^\pm)^{(0)} + L_p^2 + L_n^2. \quad (28)$$

and its eigenvalue $\omega(\omega + 4)$ is determined by the quantum number ω characterizing the $O(6)$ IRs.

(b) $\overline{SU_\pm(3)} \otimes SO(2)$ algebra. The $\overline{SU_\pm(3)} \otimes SO(2)$ algebra consists of two commuting sets of operators:

o $\overline{SU(3)}$ algebra. There are two distinct $\overline{SU_\pm(3)}$ algebras which are generated by

$$B_{ij}^{(\pm)} = G_{ij}^{(\pm)} - \frac{1}{3} \delta_{ij} M, \quad (29)$$

where

$$M = \sum_i G_{ii}^{(\pm)}. \quad (30)$$

Its second-order Casimir operator is

$$\overline{G}_3 = \sum_{ij} B_{ij}^{(\pm)} B_{ji}^{(\pm)}. \quad (31)$$

◦ $SO(2)$ algebra. The generator of this algebra is given by (30).

For $O(6) \subset U(6)$, the symmetric representation $[N]_6$ of $U(6)$ decomposes into fully symmetric $(\omega, 0, 0)_6 \equiv (\omega)_6$ IR's of $O(6)$ according to the rule [13],[16]

$$[N]_6 = \bigoplus_{\omega=N, N-2, \dots, 0(1)} (\omega, 0, 0)_6 = \bigoplus_{i=0}^{<\frac{N}{2}>} (N-2i)_6, \quad (32)$$

where $<\frac{N}{2}> = \frac{N}{2}$ if N is even and $\frac{N-1}{2}$ if N is odd. Furthermore, the following relation between the quadratic Casimir operators \overline{G}_3 of $SU(3)$, M^2 of $SO(2)$ and $C_2[O(6)]$ of $O(6)$ holds [16]:

$$C_2[O(6)] = 2\overline{G}_3 - \frac{1}{3}M^2, \quad (33)$$

which means that the reduction from $O(6)$ to the rotational group $SO(3)$ is carried out through the complementary groups $SO(2)$ and $\overline{SU(3)}$. As a consequence, the IR labels $[f_1, f_2, 0]_3$ of $SU(3)$ are determined by $(\omega)_6$ of $O(6)$ and by the integer label $(\nu)_2$ of the associated IR of $SO(2)$ i.e.

$$(\omega)_6 = \bigoplus [f_1, f_2, 0]_3 \otimes (\nu)_2. \quad (34)$$

Using the relation (33) of the Casimir operators, for their respective eigenvalues one obtains:

$$\omega(\omega+4) = \frac{4}{3}(f_1^2 + f_2^2 - f_1 f_2 + 3f_1) - \frac{\nu^2}{3}. \quad (35)$$

Thus (34) can be rewritten as [16]

$$(\omega)_6 = \bigoplus_{i=0}^{\omega} [\omega, i, 0]_3 \otimes (\omega-2i)_2 = \bigoplus_{\nu=\omega, \omega-2, \dots, 0(1)} [\omega, \frac{\omega-\nu}{2}, 0]_3 \otimes (\nu)_2,$$

or in terms of the Elliott's notation (λ, μ)

$$(\omega)_6 = \bigoplus_{\nu=\omega, \omega-2, \dots, 0(1)} \left(\frac{\omega+\nu}{2}, \frac{\omega-\nu}{2}\right) \otimes (\nu)_2. \quad (36)$$

Finally, the convenience of this reduction can be further enhanced through the use of the standard rules (11) for the reduction of the $\overline{SU(3)} \supset SO(3)$ chain.

III. THE BOSON CONDENSATE

Usually, the condensate coherent state is defined in terms of the 'condensate boson' creation operator, which is a general linear combination [10],[17]

$$B \equiv \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_M b_M, \quad (37)$$

where in general the coefficients α_i are complex. Then the (unnormalized) *condensate* coherent state is [10],[17]

$$\begin{aligned} |N; \alpha_1, \dots, \alpha_M\rangle &\propto (B^\dagger)^N |0\rangle \\ &= \left[\sum_k \alpha_k b_k^\dagger \right]^N |0\rangle, \end{aligned} \quad (38)$$

where $|0\rangle$ is the boson vacuum. The condensate coherent states (38) are often called "projective" CS. The expectation value of a one-body or two-body operator with respect to the condensate (38) can be deduced using arguments based upon formal differentiation [17].

Using the fact that the components of the two vector bosons (p_k and n_k , respectively) form a six-dimensional vector, the (unnormalized) CS for IVBM become

$$|N; \xi, \zeta\rangle \propto \left[\sum_k (\xi_k p_k^\dagger + \zeta_k n_k^\dagger) \right]^N |0\rangle, \quad (39)$$

where ξ_k and ζ_k are the components of three-dimensional complex vectors. For static problems these variables can be chosen real. The expression (39) defines the so called projective realization of the CS for the fully symmetric representation $[N]_6$ of $SU(6)$. We want to point out that in contrast to the definition of the CS for IBM-2 [18],[19],[20], where the numbers of protons, N_π , and neutrons, N_ν , are separately conserved, here only the total number of the two vector bosons $N = N_p + N_n$ is a good quantum number. The parameters ξ_k and ζ_k will determine the deformation of the nucleus in the equilibrium state.

IV. GEOMETRY

The geometry can be chosen such that $\vec{\xi}$ and $\vec{\zeta}$ to span the xz plane with the x -axis along $\vec{\xi}$ and $\vec{\zeta}$ is rotated by an angle θ about the out-of-plane y -axis, $\vec{\xi} \cdot \vec{\zeta} = r_1 r_2 \cos \theta$. In this way, the condensate can be parametrized in terms of two real coordinates r_1 and r_2 (the lengths of the two vectors), and their relative angle θ ($r_1, r_2 \geq 0$ and $0 \leq \theta \leq \pi$) in the following way

$$|N; r_1, r_2, \theta\rangle = \frac{1}{\sqrt{N!}} (B^\dagger)^N |0\rangle \quad (40)$$

with

$$B^\dagger = \frac{1}{\sqrt{r_1^2 + r_2^2}} [r_1 p_x^\dagger + r_2 (n_x^\dagger \cos \theta + n_z^\dagger \sin \theta)]. \quad (41)$$

The geometric properties of the ground states of nuclei within the framework of the IVBM can then be studied by considering the energy functional

$$E(N; r_1, r_2, \theta) = \frac{\langle N; r_1, r_2, \theta | H | N; r_1, r_2, \theta \rangle}{\langle N; r_1, r_2, \theta | N; r_1, r_2, \theta \rangle}. \quad (42)$$

By minimizing $E(N; r_1, r_2, \theta)$ (42) with respect to r_1 , r_2 , and θ , $\partial E/\partial r_1 = \partial E/\partial r_2 = \partial E/\partial \theta = 0$, one can find the equilibrium "shape" corresponding to any boson Hamiltonian, H . This shape is in many cases rigid. However, there are many situations in which the system is rather floppy and undergoes a phase transition between two different shapes.

V. SHAPE STRUCTURE OF THE DYNAMICAL SYMMETRIES

A. The $U_p(3) \otimes U_n(3)$ limit

We consider the Hamiltonian that is linear combination of first order Casimirs of $U_\tau(3)$:

$$H_I = \varepsilon_p N_p + \varepsilon_n N_n. \quad (43)$$

The Hamiltonian (43) can be rewritten in the form

$$H_I = \varepsilon_p N + \varepsilon N_n, \quad (44)$$

where $\varepsilon = \varepsilon_n - \varepsilon_p$. The first term in (44) can be dropped since it does not contribute to the energy surface. Thus, the Hamiltonian determining the properties of the system in the $U_p(3) \otimes U_n(3)$ limit is just

$$H_I = \varepsilon N_n. \quad (45)$$

The expectation value of (45) with respect to (40) gives the energy surface

$$\frac{\langle N; r_1, r_2 | H_I | N; r_1, r_2 \rangle}{\langle N; r_1, r_2 | N; r_1, r_2 \rangle} = \varepsilon \frac{N r_2^2}{r_1^2 + r_2^2}. \quad (46)$$

We see that the energy surface depends on the two parameters r_1 and r_2 and is θ -independent. In order to simplify the analysis we introduce a new dynamical variable $\rho = r_2/r_1$ as a measure of "deformation", which together with the parameter θ determine the corresponding "shape". Thus, the expression (46) becomes

$$E(N; \rho) = \varepsilon \frac{N \rho^2}{1 + \rho^2}, \quad (47)$$

which has a minimum at $\rho_0 = 0$. It corresponds to a spherical shape (vibrational limit). The scaled energy $\varepsilon(\rho) = E(N; \rho)/\varepsilon N$ in the $U_p(3) \otimes U_n(3)$ limit is given in Figure 1. The inclusion of higher-order terms in N_p and N_n will give rise to an anharmonicity.

B. The $O(6)$ limit

The Hamiltonian describing the $O(6)$ (or γ -unstable) properties can be written down through the $O(6)$ pairing operator $P^\dagger = \frac{1}{2}(p^\dagger \cdot p^\dagger - n^\dagger \cdot n^\dagger)$ in the following form

$$H_{II} = \frac{4k'}{N-1} P^\dagger P. \quad (48)$$

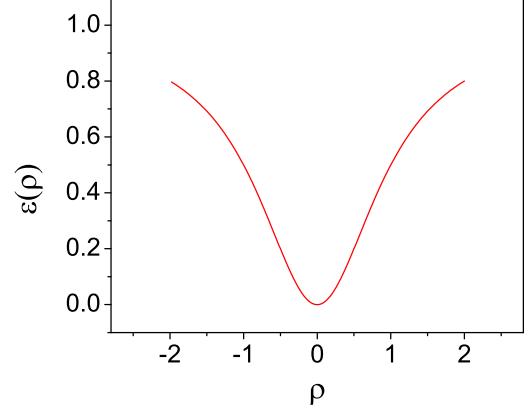


FIG. 1: (Color online) The scaled energy surface $\varepsilon(\rho)$ in the $U_p(3) \otimes U_n(3)$ limit.

In (48) the $P^\dagger P$ operator is used instead of the quadratic Casimir operator $C_2[O(6)]$ of $O(6)$ because of their linear dependence, i.e. $C_2[O(6)] = -P^\dagger P + N(N+4)$.

Taking the expectation value of (48) one obtains the energy surface

$$E(N; \rho) = k' N \left[\frac{1 - \rho^2}{1 + \rho^2} \right]^2, \quad (49)$$

which does not depend on θ (θ -unstable) and has a minimum at $\rho_0 \neq 0$ ($|\rho_0| = 1$). It corresponds to a deformed "γ-unstable" (in IBM terms) rotor. As it was mentioned, there are two $O_\pm(6)$ algebras that are isomorphic and have the same eigenspectrum but differ through phases in the wave functions resulting into different energy surfaces. The energy surface (49) corresponds to the $O_-(6)$ limit. The other, $O_+(6)$, limit is not physically important since its energy surface is just a constant. The scaled energy surface $\varepsilon(\rho)$ in the $O_-(6)$ limit is given in Figure 2.

C. The $\overline{SU}_\pm(3)$ limit

In this case we study the Hamiltonian

$$H_{III} = -\frac{k}{(N-1)} \overline{G}_3, \quad (50)$$

where \overline{G}_3 is given by (31). Note that the operators $B_{ij}^{(\omega)}$ ($\omega = \pm 1$) entering in \overline{G}_3 generate the two distinct algebras $\overline{SU}_+(3)$ and $\overline{SU}_-(3)$, respectively. The energy surface corresponding to the Hamiltonian (50) is

$$E(N; \rho, \theta) = -kN \left[\frac{2\rho^2(\cos^2 \theta + 3)}{3(1 + \rho^2)^2} + \frac{2}{3} \right], \quad (51)$$

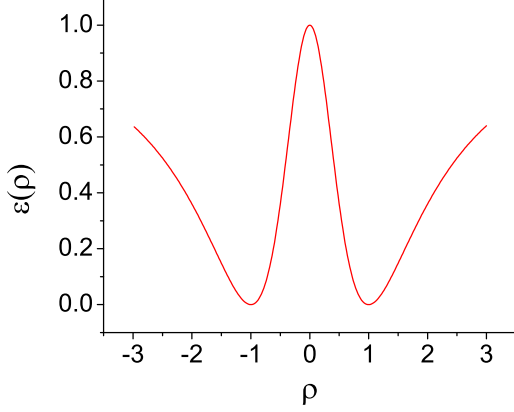


FIG. 2: (Color online) The scaled energy surface $\varepsilon(\rho)$ in the $O(6)$ limit.

for $\overline{SU_+(3)}$, and

$$E(N; \rho, \theta) = -kN \left[\frac{2\rho^2 \sin^2 \theta}{(1 + \rho^2)^2} + \frac{2}{3} \right], \quad (52)$$

for $\overline{SU_-(3)}$ algebra, respectively. We plot the scaled energy surfaces corresponding to the two limits under consideration in Figures 3 and 4, respectively. From the figures one can see that for both cases the global minimum occurs at $\rho_0 \neq 0$ ($|\rho_0| = 1$) and $\theta_0 = 0^\circ$ or $\theta_0 = 90^\circ$. The fact that the deformation parameter $|\rho| = 1$ means that we have equal deformations of the p - and n -boson subsystems which ratio is given by $\rho^2 = \frac{r_2^2}{r_1^2} = \frac{N_n}{N_p}$. Similarly, the equilibrium configurations for the combined proton-neutron $SU(3)$ and $SU^*(3)$ phase shapes in IBM-2 are obtained for $\beta_\pi = \beta_\nu$ [18],[19],[20].

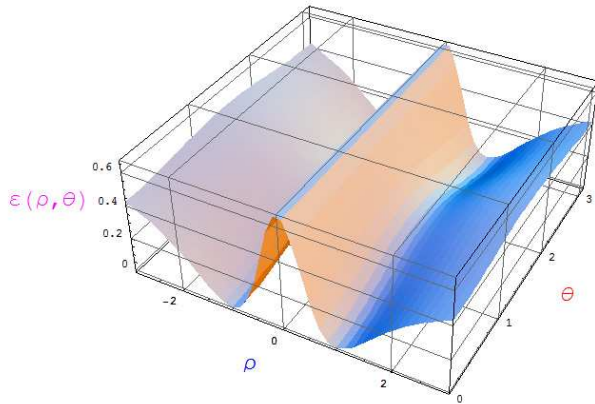


FIG. 3: (Color online) The scaled energy surface $\varepsilon(\rho, \theta)$ in the $\overline{SU_+(3)}$ limit.

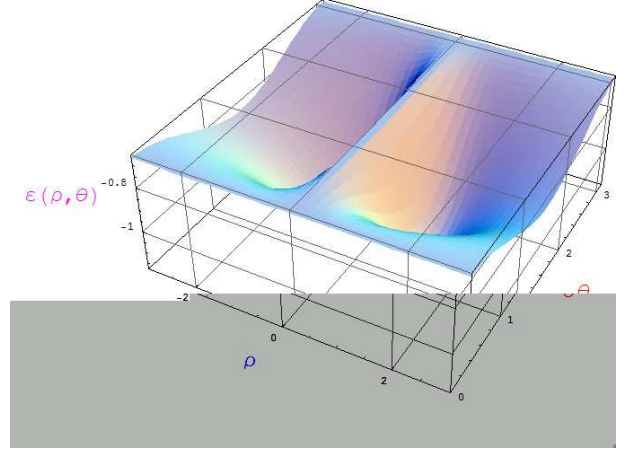


FIG. 4: (Color online) The scaled energy surface $\varepsilon(\rho, \theta)$ in the $\overline{SU_-(3)}$ limit.

D. The $SU(3) \otimes U_T(2)$ limit

This limit can be studied through the Hamiltonian

$$H_{IV} = -\frac{k}{(N-1)} K_3, \quad (53)$$

where the second order Casimir operator K_3 of $SU(3)$ is given by (21). The expectation value of (53) with respect to (42) gives the following energy surface

$$E(N; \rho, \theta) = -kN \left[\frac{(1 + 2\rho^2 \cos^2 \theta + \rho^4)}{(1 + \rho^2)^2} - \frac{2}{3} \right]. \quad (54)$$

For positive values of the parameter $k > 0$ one obtains an oscillator in the relative angle θ , which has the equilibrium at $\theta_0 = 0$. To show this one needs to consider a more general semi-classical analysis of the classical limit of H in which the complex coherent state parameters are used.

For negative values of the parameter $k < 0$, the energy surface in the $SU(3) \otimes U_T(2)$ limit is given in Figure 5. The deformed phase shape is determined through the equilibrium values $|\rho_0| = 1$ and $\theta_0 = 90^\circ$, respectively.

VI. RELATION WITH THE BOHR TRIAXIAL VARIABLE

In order to clarify to what geometry correspond the energy surfaces (potentials) obtained in the preceding sections, we consider the relation of θ with the standard triaxial "shape" parameter γ in the collective model. A relation between standard Bohr collective model shape variables used to describe the deformation of the collective motion and the shape parameters in the intrinsic state of IVBM can be obtained by calculating the expectation value of the quadrupole moments of the corresponding dynamical symmetry limit with respect to the

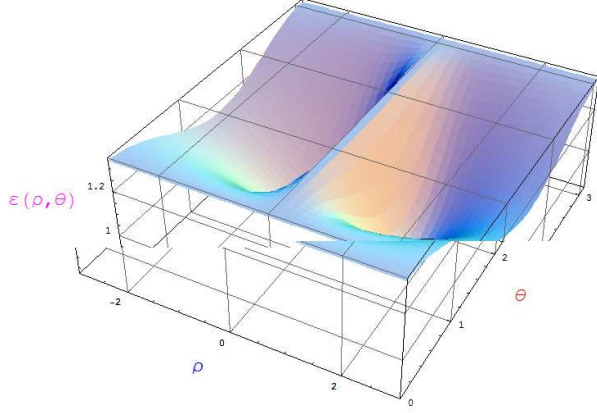


FIG. 5: (Color online) The scaled energy surface $\varepsilon(\rho, \theta)$ in the $SU(3) \otimes U(2)$ limit.

IVBM coherent state. In the intrinsic state of IVBM, the effective γ_{eff} deformation can be defined in the usual way as [21]:

$$\tan \gamma_{eff} = \sqrt{2} \frac{\langle Q_2 \rangle}{\langle Q_0 \rangle}, \quad (55)$$

where by $\langle Q_\mu \rangle$ is denoted the expectation value of the μ th component of the quadrupole operator defined for the dynamical symmetry limit under consideration.

For the $SU(3) \otimes U_T(2)$ limit one obtains:

$$\tan \gamma_{eff} = \frac{\sqrt{3}(\rho^2 \cos^2 \theta + 1)}{\left[\frac{\rho^2}{2}(-3 \cos 2\theta + 1) - 1 \right]}. \quad (56)$$

The expression (56) gives a relation between the "projective" IVBM CS deformation parameters $\{\rho, \theta\}$ and the standard collective model parameter γ_{eff} , determining the triaxiality of the nuclear system. As we saw in the previous sections, for the equilibrium values of the IVBM shape parameters $|\rho_0| = 1$ and $\theta_0 = 90^\circ$ in the $SU(3) \otimes U_T(2)$ limit one obtains $\gamma_{eff} = 60^\circ$. This corresponds to an oblate axial configuration of the compound pn -system.

Analogously, one can show that the $\overline{SU_+(3)}$ and $\overline{SU_-(3)}$ phase shapes correspond to an oblate and prolate axial deformation, respectively, of the two-fluid nuclear system.

Finally, we note that the constraint $|\rho_0| = 1$ is valid only for the limiting cases of dynamical symmetry limits. Thus for intermediate situations, one can get values for γ_{eff} different from 0° or 60° , corresponding to prolate or oblate deformed axial configuration of the nuclear system.

VII. THE GENERALIZED IVBM HAMILTONIAN AND ITS PHASE DIAGRAM

All physically interesting Hamiltonians can be combined into a single Hamiltonian that keeps all main ingredients of the considered limits. It is convenient again to scale the parameters of the two-body terms by $(N-1)$ and to consider the following Hamiltonian:

$$H = (1 - \eta)N_n + \frac{\eta}{N-1} [gK_3 + (1 - g)P^\dagger P], \quad (57)$$

The three terms in (57) correspond to the three dynamical symmetries: $U_p(3) \otimes U_n(3)$, $O(6)$ and $SU(3) \otimes U_T(2)$. We have two control parameters: η and g and hence the resulting phase diagram is two dimensional one. The phase shape diagram corresponding to the IVBM Hamiltonian (57) can be depicted as a triangle as shown in Figure 6 with each corner denoting a dynamical symmetry. For $\eta = 0$ one obtains the $U_p(3) \otimes U_n(3)$ or the *vibrational* limit; for $\eta = 1$ one encounters the two limiting cases of *deformed* shapes discussed above: $g = 0$ ($O(6)$ – γ -unstable rotor) and $g = 1$ ($SU(3)$ – axial rotor).

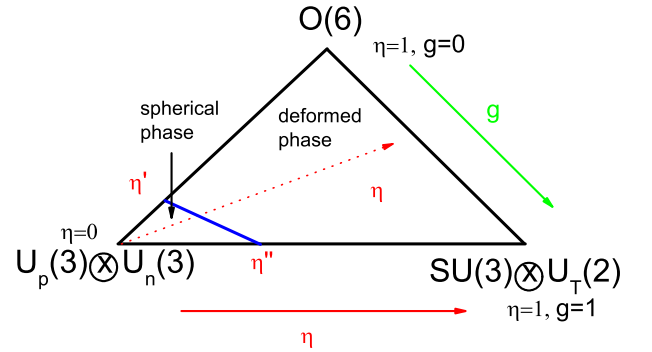


FIG. 6: (Color online) Phase diagram of IVBM. The corners of the triangle correspond to dynamical symmetries.

In some cases, the quantum phase transitions can take place between different ground state configurations or "shapes" of the system, occurring at zero temperature as a function of the corresponding control parameter. The order of the phase transitions may be determined with the standard approach (see, for instance, Ref. [2]). Here the phase transitions will be studied by analyzing the behavior of the order parameter as a function of the control parameter (Landau's approach).

It can be shown that along the leg $U_p(3) \otimes U_n(3) - O(6)$ a second order phase transition is observed at the critical value of the parameter $\eta'_c = 0.2$. For $\eta < \eta'_c$ an equilibrium spherical shape ($\rho_0 = 0$) is obtained, while for $\eta > \eta'_c$ the equilibrium shape is deformed ($|\rho_0| = 1$). The behavior of ρ_0 as a function of η is shown in Figure 7, displaying the typical behavior of a second-order transition [22]. This confirms the generic statement that for models based on $U(n_1 + n_2)$, the phase transition

between the two phases $U(n_1) \otimes U(n_2)$ and $O(n_1 + n_2)$ is of second order [23].

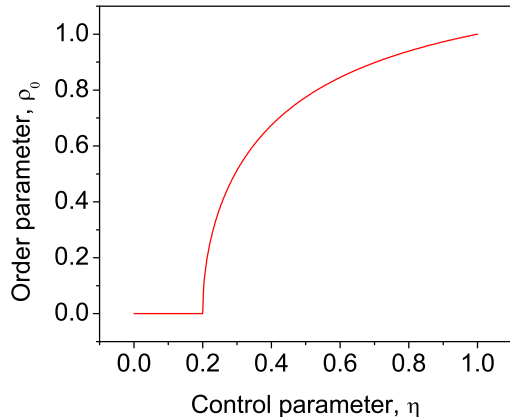


FIG. 7: (Color online) Classical order parameter ρ_0 as a function of the control parameter η for $g = 0$.

A second order phase transition between spherical and deformed axial phase shapes (the $U_p(3) \otimes U_n(3) - SU(3) \otimes U_T(2)$ leg) is observed at $\eta_c'' = 0.33$. In Figure 8 the behavior of the order parameter ρ_0 as a function of η is shown for the $U_p(3) \otimes U_n(3) - SU(3) \otimes U_T(2)$ transition. From Figure 8 one can see again the typical behavior of a second-order transition [22].

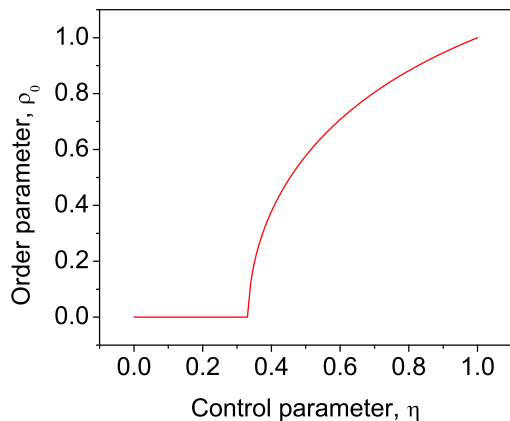


FIG. 8: (Color online) Classical order parameter ρ_0 as a function of the control parameter η for $g = 1$.

Taking the expectation value of the more general Hamiltonian (57) one obtains its classical limit (Landau potential) in the following form:

$$\varepsilon(\rho, \theta) = \frac{\varepsilon_0 + A(\theta)\rho^2 + C\rho^4}{(1 + \rho^2)^2}, \quad (58)$$

where $A(\theta) = 1 + \eta[2g(1 + \cos^2 \theta) - 3]$, $C = 1$, and $\varepsilon_0 = \eta$. The potential (58) has a generic Landau-like

form, except that the cubic term is missing. Note also that, in contrast to the traditional Landau theory, the denominator in (58) ensures a finite potential when $\rho \rightarrow \infty$. The minimization in the variable θ can be performed separately since the dependence of the potential (58) is only through the $\cos^2 \theta$. This potential has a minimum either at $\theta_0 = 0^\circ$ or $\theta_0 = 90^\circ$ and setting to these values we can study only the ρ dependence. The global minimum of the potential (58) can be found by its minimization with respect to ρ . If such a minimum occurs at $(\rho = 0)$ one has the higher (spherical) symmetry and for $(\rho \neq 0)$, the lower (deformed) symmetry. The minimization gives two solutions

$$\rho_0 = 0, \quad (59)$$

and (setting $\theta_0 = 90^\circ$)

$$\rho_0 = \pm \sqrt{\frac{1 - 5\eta + 2g\eta}{-1 - 3\eta + 2g\eta}}. \quad (60)$$

When $g = 0$ or $g = 1$, we recover the familiar results for the critical point values $\eta_c = 0.2$ or $\eta_c = 0.33$ at which a phase transition is observed from spherical to γ -unstable deformed or from spherical to axially deformed phase shape. The critical value η_c at which the phase transition appears moves from 0.2 to 0.33 with the increase of g from 0 to 1.

For $\eta = 1$, we obtain a constant value of $\rho_0 = 1$ as a function of g (the $O(6) - SU(3) \otimes U_T(2)$ leg of Fig. 6), which indicates that no phase transition occurs. For a fixed value of $\eta < 1$ the equilibrium value of ρ_0 decreases smoothly. In Figure 9, the behavior of the order parameter ρ_0 as a function of the control parameter g is shown for $\eta = 0.5$.

For $\eta = 0.33$ a second order phase transition occurs from deformed to spherical phase shape at the critical point $g = 1$, which moves to the lower values of g with the further decrease of η (> 0.2). This phase transition as a function of g disappears at $\eta = 0.2$. In Figure 10, the behavior of the order parameter ρ_0 as a function of the control parameter g is shown for $\eta = 0.3$.

We note that due to the missing of the cubic term in (58) the first order phase transitions do not appear. Hence the spherical-deformed second order phase transition occurs along the continuous line $\eta' - \eta''$, but not at isolated points. Indeed, from Eq.(60) one easily obtains the critical line equation

$$\eta = \frac{1}{5 - 2g}, \quad (61)$$

which reduces to the two limiting critical points $\eta' = 0.2$ ($g=0$) and $\eta'' = 0.33$ ($g=1$), respectively.

VIII. SUMMARY

In the present paper, the geometrical analysis of the different dynamical symmetries of the two-fluid IVBM

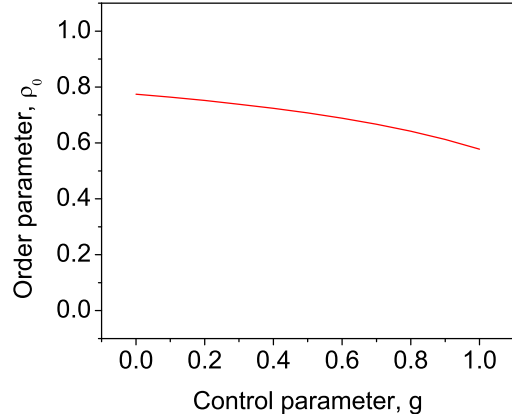


FIG. 9: (Color online) Classical order parameter ρ_0 as a function of the control parameter g for $\eta = 0.5$.

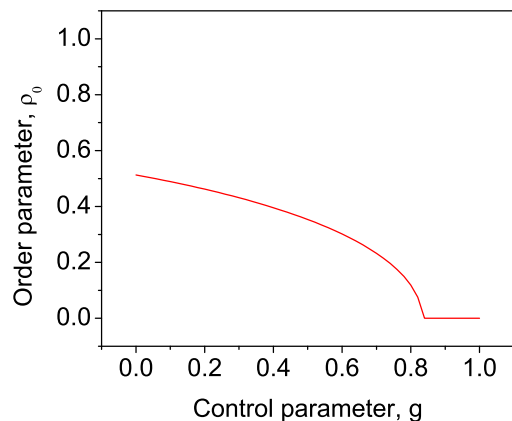


FIG. 10: (Color online) Classical order parameter ρ_0 as a function of the control parameter g for $\eta = 0.3$.

with the $U(6)$ as a dynamical group is carried out by means of coherent state method. The latter allows the calculation of the classical limit of the Hamiltonians corresponding to different dynamical symmetries in terms of appropriately chosen classical (geometrical) variables representing the boson degrees of freedom. The different dynamical symmetries correspond to qualitatively distinct ground state equilibrium configurations, which constitute the phases of the system. Thus, the dynamical symmetries are the ones that determine the structure of the corresponding phase diagram.

We have studied the phase structure of IVBM and three phase shapes corresponding to its three dynamical limits have been obtained: spherical, $U_p(3) \otimes U_n(3)$, γ -unstable, $O(6)$, and axially deformed shape, $SU(3) \otimes U_T(2)$. Further, the ground state quantum phase transitions between different phase shapes, corresponding to the different dynamical symmetries and mixed symme-

try case, are investigated theoretically using one or (for the more general IVBM Hamiltonian (57)) two control parameters in the Hamiltonian. The latter drive the system in different phases. It is shown that a second order phase transition occurs from spherical to deformed phase shape.

In conclusion, we concern the question of the possible physical interpretation of the introduced classical variables. In general, there is a variety of ways in which the geometrical variables can be defined. Under some constraints different realizations of the CS can be related to each other. In practice, it turns out that the most convenient are the "projective" coherent states, which for the IVBM were defined by (39). The two set of dipole variables $\{\xi_k\}$ and $\{\zeta_k\}$ entering in (39) can be related with the quadrupole-octupole deformations realized in many regions of nuclear chart. The presence of octupole deformation causes a shift of the nuclear center of mass which must be balanced by addition of a dipole deformation, which in lowest order is proportional to the product of quadrupole and octupole deformations.

In our opinion the physical meaning should be associated with the physical quantities which can be constructed from the "projective variables" rather than these variables themselves. This can be easily seen if the "algebraic" CS [2] are used. The introduction of the algebraic CS (and the respective "algebraic geometrical variables") is based on the precise mathematical procedure related with the theory of the coset spaces, which allows one to attach a geometrical space to a certain algebra g . Geometrical variables are then associated with the elements of the considered coset space. In this regard, the IVBM turns out to be a particular case of a more general class of algebraic models, called two-level pairing models [23]. An associated geometry for the two-level pairing models is defined through the coset space $U(n_1+n_2)/U(n_1) \otimes U(n_2)$ (where $n_1 = 2L_1 + 1$ and $n_2 = 2L_2 + 1$), which obviously generalizes the coset space of the usual two-level boson models for which $n_1 = 1$ (i.e. one of the bosons is a scalar boson, s). In our case we have $n_1 = n_2 = 3$ which leads to a much richer phase space in 9-dimensions, including in addition to the quadrupole also monopole and dipole collective degrees of freedom.

Finally, one might relate the "projective variables" with the cluster degrees of freedom in nuclei, connected with the relative motion of the clusters. The cluster degrees of freedom as well as the octupole ones play an important role in the description of the negative parity states in nuclei.

Acknowledgments

This work was supported by the Bulgarian National Foundation for scientific research under Grant Number DID-02/16/17.12.2009. The author is grateful to Alberto Ventura and Ana Georgieva for reading the manuscript.

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- [1] F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000).
 - [2] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
 - [3] F. Iachello and R. D. Levine, *Algebraic Theory of Molecules* (Oxford: Oxford University Press, 1995).
 - [4] S. Kuyucak, Chem. Phys. Lett. **301**, 435 (1999); F. Perez-Bernal, L. F. Santos, P. H. Vaccaro, and F. Iachello, Chem. Phys. Lett. **414**, 398 (2005).
 - [5] H. Yepez-Martinez, J. Cseh, and P. O. Hess, Phys. Rev. C **74**, 024319 (2006).
 - [6] R. Gilmore, J. Math. Phys. **20**, 891 (1979).
 - [7] A. E. L. Dieperink, O. Scholten, F. Iachello, Phys. Rev. Lett. **44**, 1747 (1980).
 - [8] O. S. van Roosmalen, R.D. Levine, A.E.L. Dieperink, Chem. Phys. Lett. **101**, 512 (1983).
 - [9] W. M. Zhang, D. H. Feng and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
 - [10] P. Cejnar and J. Jolie, Progr. Part. Nucl. Phys. **62**, 210 (2009).
 - [11] A. I. Georgieva, P. Raychev, and R. Rouseev, J. Phys. G: Nucl. Phys. **8**, 1377 (1982).
 - [12] A. I. Georgieva, H. G. Ganev, J. P. Draayer and V. P. Garistov, Phys. Part. Nucl. **40**, 469 (2009).
 - [13] V. V. Vanagas, *Algebraic methods of microscopic nuclear theory*, (Mintis, Vilnius, 1971) (in russian).
 - [14] D. Kusnezov, Phys. Rev. Lett. **79**, 537 (1997).
 - [15] A. M. Shirokov, N. A. Smirnova, and Yu. F. Smirnov, Phys. Lett. B **434**, 237 (1998).
 - [16] H. G. Ganev, A. I. Georgieva and J. P. Draayer, Phys. Rev. C **72**, 014314 (2005).
 - [17] P. Van Isacker and J.-Q. Chen, Phys. Rev. C **24**, 684 (1981); M. A. Caprio, J. Phys. A: Math. Gen. **38**, 6385 (2005).
 - [18] A. E. L. Dieperink, Nucl. Phys. **A421**, 189c (1984).
 - [19] M. A. Caprio and F. Iachello, Phys. Rev. Lett. **93**, 242502 (2004).
 - [20] J.M. Arias, J. E. Garcia-Ramos, and J. Dukelsky, Phys. Rev. Lett. **93**, 212501 (2004).
 - [21] A. Bohr and B. R. Mottelson, Nuclear Structure, Vol. II., (W. A. Benjamin Inc., New York, 1975).
 - [22] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford Science, New York, 1971).
 - [23] M. A. Caprio, P. Cejnar, and F. Iachello, Ann. Phys **323**, 1106 (2008).